

Due Fri

1.3 – Matrices and Matrix Operations

Definition 1: A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The **size** of a matrix that has m rows and n columns is $m \times n$ (read “ m by n ”).

6. Use the following matrices to compute the indicated expression if it is defined.

$$\begin{matrix}
 3 \times 2 & & 2 \times 2 & & 2 \times 3 \\
 A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, & B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, & C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \\
 D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, & E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}
 \end{matrix}$$

- a. $(2D^T - E)A$
- b. $(4B)C + 2B$
- c. $(-AC)^T + 5D^T$
- d. $(BA^T - 2C)^T$
- e. $B^T(CC^T - A^T A)$
- f. $D^T E^T - (ED)^T$

$$\text{tr}(E) = 6 + 1 + 3 = 10$$

Definition 7: If A is any $m \times n$ matrix, then the **transpose of A** , denoted by A^T , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A ; that is, the first column of A^T is the first row of A , the second column of A^T is the second row of A , and so forth.

$$D^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \quad \swarrow \text{Scalar multiplication} \quad 2D^T = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix}$$

$$2D^T - E = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

Add/subt. are performed entry by entry,
so the matrices must be the same size.

$$(2D^T - E)A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

3×3 3×2 — size of the result
 These #s must match

$$= \begin{bmatrix} -4(3) - 3(-1) + 3(1) & -4(0) - 3(2) + 3(1) \\ 11(3) - 1(-1) + 2(1) & 11(0) - 1(2) + 2(1) \\ 0(3) + 1(-1) + 5(1) & 0(0) + 1(2) + 5(1) \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

3×2

This process is called the row-column rule.

Relabeling: $A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$

row vectors of A are $\vec{r}_1 = [-4 \ -3 \ 3]$

$$\vec{r}_2 = [11 \ -1 \ 2]$$

$$\vec{r}_3 = [0 \ 1 \ 5]$$

column vectors of B are $\vec{c}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

AB can also be found using the column method:

$$AB = [A\vec{c}_1 \mid A\vec{c}_2]$$

This is a partitioned matrix.

$$A\vec{c}_1 = \begin{matrix} \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} & \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} & = & \begin{bmatrix} -6 \\ 36 \\ 4 \end{bmatrix} \\ 3 \times 3 & 3 \times 1 & & 3 \times 1 \end{matrix}$$

OR $AB = \begin{bmatrix} \vec{r}_1 & B \\ \vec{r}_2 & B \\ \vec{r}_3 & B \end{bmatrix}$

this is the row method.

17. Use the column-row expansion of AB to express this product as a sum of matrix products.

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

The **column-row expansion** of AB is $AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$, where \mathbf{c}_i are column vectors of A and \mathbf{r}_i are row vectors of B .

$$\begin{aligned} AB &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & -9 & -3 \\ 2 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -5 & 5 \\ 2 & -1 & 3 \end{bmatrix} \end{aligned}$$

$$A\vec{x} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix} x_3$$

entries of x

columns of A

Definition 6: If A_1, A_2, \dots, A_r are matrices of the same size, and if c_1, c_2, \dots, c_r are scalars, then an expression of the form $c_1 A_1 + c_2 A_2 + \dots + c_r A_r$ is called a **linear combination** of A_1, A_2, \dots, A_r with **coefficients** c_1, c_2, \dots, c_r .

22. Example 6 of section 1.2 presents the linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

and the reduced row echelon form of its associated augmented matrix as

$$\left[\begin{array}{cccccc|c} \underline{1} & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & \underline{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ x_6 = 0 \end{array}$$

Let $r = x_2, s = x_4, t = x_5$

Express the solution as a linear combination of column vectors that contain only numerical entries.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t$$

Theorem 1.3.1 If A is an $m \times n$ matrix, and if \mathbf{x} is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of \mathbf{x} .

The **main diagonal** of a square matrix contains entries from the upper left to lower right corners.

Definition 8: If A is a square matrix, then the **trace of A** , denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.